

A characterization of $\mathrm{PSL}(2, q)$, $q = 5, 7$

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Abstract

In this short note we prove that the finite non-abelian simple groups $\mathrm{PSL}(2, q)$, where $q = 5, 7$, are determined by their posets of classes of isomorphic subgroups. In particular, this disproves the conjecture in the end of [5].

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Key words: finite simple groups, posets of isomorphic subgroups.

1 Introduction

In group theory there are many ways to recognize the finite simple groups: by spectrum, by prime graph, by non-commuting graph, by subgroup lattices, ... and so on. Another way to recognize the first two non-abelian simple groups, $\mathrm{PSL}(2, q)$ with $q = 5, 7$, is presented in the following. It uses the poset $\mathrm{Iso}(G)$ of classes of isomorphic subgroups of a group G (see [5]):

$$\mathrm{Iso}(G) = \{[H] \mid H \leq G\}, \text{ where } [H] = \{K \leq G \mid K \cong H\}.$$

Recall that $\mathrm{Iso}(G)$ is partially ordered by

$$[H_1] \leq [H_2] \text{ if and only if } K_1 \subseteq K_2 \text{ for some } K_1 \in [H_1] \text{ and } K_2 \in [H_2].$$

Obviously, all finite abelian simple groups G have the same poset $\mathrm{Iso}(G)$ (a chain of length 1) and consequently they cannot be recognized in this way. In the non-abelian case the situation is better, as shows our main result.

Theorem. *Let $G_0 \in \{\mathrm{PSL}(2, 5), \mathrm{PSL}(2, 7)\}$ and G be a finite group such that $\mathrm{Iso}(G) \cong \mathrm{Iso}(G_0)$. Then $G \cong G_0$.*

This leads to a natural question.

Question. Let G_0 be a finite non-abelian simple group and G be a finite group such that $\text{Iso}(G) \cong \text{Iso}(G_0)$. Is it true that $G \cong G_0$?

2 Proof of the main results

We start with the following easy but important lemma.

Lemma. *Let G_0, G be two finite groups, $f : \text{Iso}(G_0) \longrightarrow \text{Iso}(G)$ be a poset isomorphism and H_0, H be two subgroups of G_0 and G , respectively, such that $f([H_0]) = [H]$. Then:*

- a) $\text{Iso}(H_0) \cong \text{Iso}(H)$.
- b) *If $|H_0| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_i, i = 1, 2, \dots, k$, are distinct primes, then $|H| = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$ for some distinct primes q_1, q_2, \dots, q_k .*
- c) *If all isomorphic copies of H_0 are maximal subgroups of G_0 , then H is a maximal subgroup of G .*

Proof. a) It is obvious that f induces a poset isomorphism from $\text{Iso}(H_0)$ to $\text{Iso}(H)$.

b) This follows from a) and Theorem 3.2 of [5].

c) Let K be a subgroup of G such that $H \subset K \subset G$. Then $[H] < [K] < [G]$ and therefore $f^{-1}([H]) < f^{-1}([K]) < f^{-1}([G])$, i.e. $[H_0] < [K_0] < [G_0]$ where $[K_0] = f^{-1}([K])$. It follows that there are $H'_0 \cong H_0$ and $K'_0 \cong K_0$ such that $H'_0 \subset K'_0 \subset G_0$, a contradiction. ■

Remark. The assumption in c) of the above lemma is justified, because a subgroup M' isomorphic to a maximal subgroup M of a group G is not necessarily maximal. For example, let G be a finite non-abelian simple group, $M = \{(x, x) | x \in G\}$ and $M' = G \times 1$. Then M is maximal in G , $M' \cong M$ ($\cong G$), but clearly M' is not maximal.

We are now able to prove our main result.

Proof of the main theorem. Assume first that $G_0 = \text{PSL}(2, 5)$. By the above lemma we have $|G| = p^2qr$, where p, q, r are distinct primes.

It is well-known that the maximal subgroups of $\text{PSL}(2, 5)$ are of order 12 (isomorphic with A_4), 10 (isomorphic with D_{10}), and 6 (isomorphic with S_3). Moreover, any subgroup of $\text{PSL}(2, 5)$ isomorphic with A_4 , D_{10} or S_3 is also maximal. Therefore, if $f : \text{Iso}(G_0) \longrightarrow \text{Iso}(G)$ is a poset isomorphism and $[M_1] = f([A_4])$, $[M_2] = f([D_{10}])$ and $[M_3] = f([S_3])$, then M_1 , M_2 and M_3 are maximal subgroups of G of order p^2q (or p^2r), pq and pr , respectively. Suppose now that G contains a maximal subgroup M which is not isomorphic with M_1 , M_2 or M_3 . Then $[M] < [M_1]$, $[M] < [M_2]$ or $[M] < [M_3]$ because $[M_1]$, $[M_2]$ and $[M_3]$ are the maximal elements of $\text{Iso}(G)$. This implies that $|M|$ is a proper divisor of $|M_1|$, $|M_2|$ or $|M_3|$, i.e. $|M| \in \{p, q, r, pq \text{ (or } pr)\}$. Consequently, the orders of maximal subgroups of G are p^2q (or p^2r), pq , pr , and possibly proper divisors of these numbers.

Then G has no Sylow system (it cannot have subgroups of order qr) and therefore it is not solvable. Since $\text{PSL}(2, 5) \cong A_5$ is the unique non-solvable group of order p^2qr (see e.g. [1]), it follows that $G \cong G_0$, as desired.

Assume next that $G_0 = \text{PSL}(2, 7)$. Then $|G| = p^3qr$, where p, q, r are distinct primes. Since the maximal subgroups of $\text{PSL}(2, 5)$ are of order 24 (isomorphic with S_4) and 21 (isomorphic with the Frobenius group of order 21), a similar argument implies that the orders of maximal subgroups of G are p^3q (or p^3r), qr , and possibly proper divisors of these numbers.

Again, G has no Sylow system (it cannot have subgroups of order qr) and thus it is not solvable. This shows that G has a composition factor, say G_1/G_2 , which is a non-abelian simple group. Then $|G_1/G_2|$ is even. Moreover, Theorem 1.35 of [3] implies that it is divisible by 4. This leads to $p = 2$, i.e. $|G| = 8qr$. Consequently, by [2], pp. 12-14, we have either $G_1/G_2 \cong \text{PSL}(2, 5)$ or $G_1/G_2 \cong \text{PSL}(2, 7)$. In the first case we infer that G has order 120 and so it is isomorphic to one of the following groups: S_5 , $A_5 \times \mathbb{Z}_2$, and $\text{SL}(2, 5)$. This is impossible because all these groups have no maximal subgroup of order 15. Then $G_1/G_2 \cong \text{PSL}(2, 7)$, which shows that $G \cong G_0$. This completes the proof. ■

References

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